AP Calculus AB — Homework 0303 - Solutions
Dr. Paul L. Bailey — Monday, March 3, 2025

Problem 6 (Thomas §7.4 # 90a)**.** Find the equation of the line through the origin and tangent to the graph of $y = \ln(x)$.

Solution. Let $f(x) = \ln(x)$ so that $f'(x) = \frac{1}{x}$. Let $(a, f(a))$ denote the point of tangency. Then the linearization of f at a is

$$L(x) = \frac{1}{a}(x - a) + \ln(a) = \frac{x}{a} + (\ln(a) - 1).$$

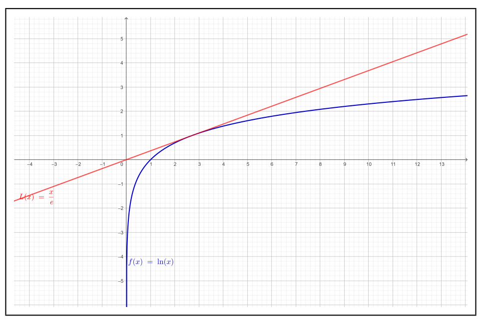
We wish to find a so that this passes through the origin, which means the y-intercept is zero, which implies that $\ln(a) = 1$, so $a = e$. Thus $L(x) = \frac{x}{e}$ and the tangent line is

$$y = \frac{x}{e}.$$

□

Problem 7 (Thomas §7.4 # 90b)**.** Show that $\ln(x^e) < x$ for all positive $x \neq e$.

Solution. Let $f(x) = \ln(x)$ and $L(x) = \frac{x}{e}$. We have $f'(x) = \frac{1}{x}$ and $f''(x) = -\frac{1}{x^2} < 0$ for all $x > 0$. Thus the f is concave down, so the entire curve $y = \ln(x)$ lies below the tangent line except at the point of tangency.



The point of tangency is $(e, 1)$. For $x \neq e$, we have

$$L(x) > f(x) \Rightarrow \frac{x}{e} > \ln(x) \Rightarrow x > e\ln(x) = \ln(x^e).$$

This shows that $\ln(x^e) < x$ for all positive $x \neq e$. Show that $\ln(x^e) < x$ for all positive $x \neq e$. □

Problem 8 (Thomas §7.4 # 90c)**.** Show that $x^e < e^x$ for all positive $x \neq e$. Conclude that $\pi^e < e^\pi$.

Solution. By the previous problem, $\ln(x^e) < x$ for all positive $x \neq e$.

Since exp is an increasing function, taking exp of both sides yields

$$\exp(\ln(x^e)) < \exp(x);$$

that is,

$$x^e < e^x$$

for all positive $x \neq e$. Since $\pi \neq e$, we obtain

$$\pi^e < e^\pi.$$

□